§2.6 Exact D.E.

Recall: Separable

$$M(y)\,y' = N(x)$$

$$\int M(y)\,dy = \int M(y)\,y'\,dx = \int N(x)\,dx$$

Linear

$$y' + py = g$$

$$\mu y = \int (\mu y)'\,dx = \int \mu g\,dx$$

Exact

$$M(x,y)\,y' + N(x,y) = 0$$

$$\underbrace{\int M(x,y)\,y' + N(x,y)\,dx}_{??} = \int 0\,dx \rightsquigarrow C$$

This integral exists if

$$\frac{\partial}{\partial x}\big(f(x,y)\big) = M(x,y)\,y' + N(x,y)$$

$\Rightarrow$ $f(x,y)$ is potential function for v.f. $M(x,y)\,dy + N(x,y)\,dx$

$\rightsquigarrow$ Reformulation:

$$M(x,y)\,y' + N(x,y) = 0$$

$$\Downarrow ?$$

$$M(x,y)\,dy + N(x,y)\,dx = 0$$

Integrate: $\underbrace{\int M(x,y)\,dy + N(x,y)\,dx}_{\text{potential function.}} = C$

$\Rightarrow$ Vector Field MUST be conservative !!

$$\frac{\partial}{\partial x} M(x,y) = \frac{\partial}{\partial y} N(x,y)$$

Methods for finding potential function:

1. $f(x,y) =$ "least common sum" of $\begin{cases} \int M(x,y)\,dy \\ \int N(x,y)\,dx \end{cases}$

2. v.f. world $\qquad$ function world

$M(x,y)\,dy \rightsquigarrow$ partial answer $f^y = \int M(x,y)\,dy$

$N(x,y)\,dx - \frac{\partial}{\partial x} f^y\,dx \rightsquigarrow$ remaining terms $f^x = \int N(x,y) - \frac{\partial}{\partial x} f^y\,dx$

$$\boxed{f = f^x + f^y}$$

Exact DE (Conservative v.f)

$$M(x,y)\,y' + N(x,y) = 0$$

where $\partial/\partial x\, M = \partial/\partial y\, N$

solution: $f(x,y)=c$ where $f = \text{lcs}\begin{pmatrix}\int M\,dy \\ \int N\,dx\end{pmatrix}$

EX: $2xy\,y' = -(2x+y^2)$

$$\underbrace{2xy}_{\partial/\partial x \,\to\, 2y}\,y' + \underbrace{(2x+y^2)}_{\partial/\partial y \,\to\, 2y} = 0$$

$2y \overset{\checkmark}{=} 2y$ Exact!

$$\int 2xy\,dy + (2x+y^2)\,dx = c$$

$\int 2xy\,dy = xy^2$

$\int 2x+y^2\,dx = x^2 + xy^2$

$$\boxed{x^2 + xy^2 = c}$$

EX: $(1+2xy)\,y' + (y^2+2x) = 0$

$$\underbrace{(1+2xy)}_{\partial/\partial x \,\to\, 2y}\,y' + \underbrace{(y^2+2x)}_{\partial/\partial y \,\to\, 2y} = 0$$

$2y \overset{\checkmark}{=} 2y$ Exact!

$$\int (1+2xy)\,dy + (y^2+2x)\,dx = c$$

$\int 1+2xy\,dy = y + xy^2$

$\int y^2+2x\,dx = xy^2 + x^2$

$$\boxed{y + xy^2 + x^2 = c}$$

EX: $(y\cos x + 2xe^y)\,dx + (\sin x + x^2e^y - 1)\,dy = 0$

$\partial/\partial y$: $\cos x + 2xe^y \overset{\checkmark}{=} \cos x + 2xe^y$:$\partial/\partial x$ EXACT!

$dy \to y'$

$$\int (y\cos x + 2xe^y)\,dx + (\sin x + x^2e^y - 1)\,dy = 0$$

$\int y\cos x + 2xe^y\,dx = y\sin x + x^2e^y$

$\partial/\partial y$

$\int \cancel{\sin x} + \cancel{x^2e^y} - 1\,dy = y\sin x + x^2e^y - y$

$$\boxed{y\sin x + x^2e^y - y = c}$$

EX: Sometimes lcs is unpleasant...

$$(x+2e^{2x}x^3y+5e^{2x}x^2y+2e^{2x}xy)\,dx+(e^{2x}x^3+e^{2x}x^2+1)\,dy=0$$

$\downarrow \partial/\partial y$ $\qquad\qquad$ $\downarrow \partial/\partial x$

$$2e^{2x}x^3+5e^{2x}x^2+2e^{2x}x \overset{\checkmark}{=} 2e^{2x}x^3+3e^{2x}x^2+2e^{2x}x^2+2e^{2x}x$$

$$\int x+\cancel{2e^{2x}x^3y}+\cancel{5e^{2x}x^2y}+\cancel{2e^{2x}xy}\,dx=\underline{\text{this}}+\tfrac{1}{2}x^2$$

$$\cancel{2e^{2x}x^3y}+\cancel{5e^{2x}x^2y}+\cancel{2e^{2x}xy} \xleftarrow{\partial/\partial x}$$

$$\int e^{2x}x^3+e^{2x}x^2+1\,dy = e^{2x}x^3y+e^{2x}x^2y+y$$

$$\boxed{e^{2x}x^3y+e^{2x}x^2y+y+\tfrac{1}{2}x^2=C}$$

Integrating Factors for Exact DE:

→ Sometimes you can force DE to be exact!

EX $\quad (x^2+xy)y'+(3xy+y^2)=0$

$\downarrow \partial/\partial x \qquad \downarrow \partial/\partial y$

$2x+y \neq 3x+2y$ Not exact!!

→ Try to solve $\int (x^2+xy)\,dy+(3xy+y^2)\,dx$

$$\int x^2+xy\,dy = x^2y+\tfrac{1}{2}xy^2$$

$$\int 3xy+y^2\,dx = \tfrac{3}{2}x^2y+xy^2$$

These should have matched...

Modify: Multiply by $\mu = x$

$$x(x^2+xy)y'+x(3xy+y^2)=0$$

$$(x^3+x^2y)y'+(3x^2y+xy^2)=0$$

$\downarrow \partial/\partial x \qquad \downarrow \partial/\partial y$

$3x^2+2xy \overset{\checkmark}{=} 3x^2+2xy$ EXACT

$$\int (x^3+x^2y)\,dy+(3x^2y+xy^2)\,dx=C$$

$$\int x^3+x^2y\,dy = x^3y+\tfrac{1}{2}x^2y^2$$

$$\int 3x^2y+xy^2\,dx = x^3y+\tfrac{1}{2}x^2y^2$$

$$\boxed{x^3y+\tfrac{1}{2}x^2y^2=C}$$

Integrating Factor

$$M(x,y)\,y' + N(x,y) = 0$$

$$\Downarrow$$

$$\mu M y' + \mu N = 0$$

$$\partial/\partial x\,(\mu M) = \partial/\partial y\,(\mu N)$$

(Hope μ is very simple
- function w/ only x $\quad \mu = \mu(x)$
- function w/ only y $\quad \mu = \mu(y)$)

$\mu = \mu(x)$

$$\partial/\partial x\,(\mu M) = \mu' M + \mu M_x$$

$$\partial/\partial y\,(\mu N) = \mu N_y$$

$$\mu' M + \mu M_x = \mu N_y$$

$$\mu' = \mu\left(\frac{N_y - M_x}{M}\right) \quad \text{(separable)}$$

$$\int \frac{1}{\mu}\,d\mu = \int \frac{N_y - M_x}{M}\,dx$$

This must be a function of only x

$$\mu = e^{\int \frac{N_y - M_x}{M}\,dx}$$

Formulas:

- If $\frac{N_y - M_x}{M}$ function of x

$$\mu = e^{\int \frac{N_y - M_x}{M}\,dx}$$

- If $\frac{M_x - N_y}{N}$ function of y

$$\mu = e^{\int \frac{M_x - N_y}{N}\,dy}$$

EX: $\quad y + (2xy - e^{-2y})\,y' = 0$

$N = y \xrightarrow{\partial/\partial y} 1$ $\qquad M = 2xy - e^{-2y} \xrightarrow{\partial/\partial x} 2y$

$1 \neq 2y$ $\qquad$ Not exact.

compare $1-2y$ to $\begin{cases} y \\ 2xy - e^{-2y} \end{cases}$

$\rightarrow \frac{1-2y}{y}$ is function w/ only y

$$\mu = e^{\int \frac{2y-1}{y}\,dy} = e^{2y - \ln y} = \boxed{\frac{e^{2y}}{y}}$$

DE becomes: $e^{2y} + \left(2xe^{2y} - \frac{1}{y}\right)y' = 0$